

MAS231 Geometry II

Test

21 February 2007

The duration of this test is 45 minutes. Write your name and student number in the spaces below.

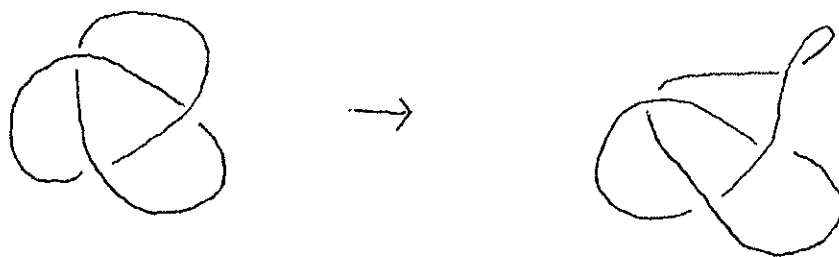
Electronic calculators may be used. The calculator must not be preprogrammed prior to the examination. Enter the name and type here _____

Answer all questions. Write all answers in the spaces provided. If you run out of space for an answer, continue on the back of the page.

Name: _____

Student Number: _____

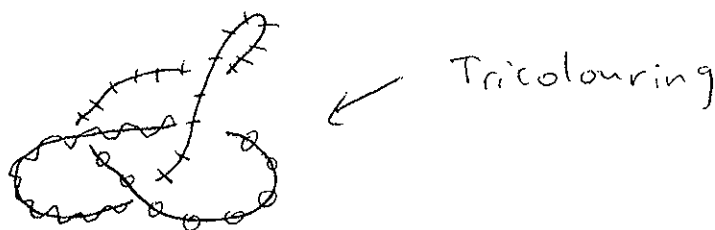
1.



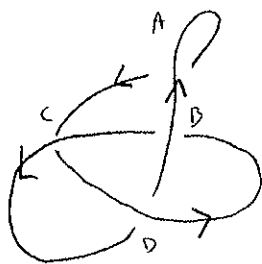
- (a) 5marks Which Reidemeister move is used in the transition between the two knot diagrams above?

Reidemeister Move I

- (b) 7marks Either find a tricolouring of the knot diagram on the right in the two knot diagrams above or prove that no tricolouring exists.



- (c) 8marks Find the writhe of the knot diagram on the right in the two knot diagrams above. Do you expect it to be the same as the writhe of the knot diagram on the left in the two knot diagrams above?



$$\text{sgn}(A) = +1$$

$$\text{sgn}(B) = +1$$

$$\text{sgn}(C) = +1$$

$$\text{sgn}(D) = +1$$

$$\text{Writhe} = 1 + 1 + 1 + 1 = 4$$

writhe of knot on left will be different
because writhe is not invariant under
Type I moves.

2.

- (a)
- 6marks
- State the inductive definition of the Kauffman bracket
- B
- .

$$1. B(\bigcirc) = 1$$

$$2. B(\text{positive crossing}) = x B(\text{smooth}) + x^{-1} B(\text{negative crossing})$$

$$B(\text{negative crossing}) = x^{-1} B(\text{smooth}) + x B(\text{positive crossing})$$

$$3. B(\square \bigcirc) = -(x^2 + x^{-2}) B(\square)$$

- (b)
- 7marks
- Find the Kauffman bracket of the knot diagram



$$B(\text{diagram}) = B(\bigcirc) \quad (\text{by applying Reid. Move II})$$

$$= 1$$

- (c)
- 7marks
- Using the result in part (b), find the Kauffman Bracket of this link:



$$B(\text{link}) = -(x^2 + x^{-2}) B(\text{diagram})$$

$$= -(x^2 + x^{-2}) (1)$$

$$= -(x^2 + x^{-2})$$

3. 6marks Define the Jones polynomial $J(t)$ of a knot diagram in terms of its writhe W and Kauffman Bracket. State what is the Jones polynomial $J_R(t)$ of the mirror image of a knot in terms of the Jones polynomial $J(t)$ of the knot itself.

$$J(x) = (-x)^{3W} B(x)$$

$J(t)$ is obtained by
setting $t = x^4$

$$J_R(t) = J(t^{-1})$$

4. 6marks Suppose that $\gamma(t)$ and $\mu(t)$ are curves in $\mathbb{R}^3$ and define $\dot{\gamma}(t) = \frac{d}{dt}\gamma(t)$ and $\dot{\mu}(t) = \frac{d}{dt}\mu(t)$. Prove that

$$\frac{d}{dt}(\gamma(t) \cdot \mu(t)) = \dot{\gamma}(t) \cdot \mu(t) + \gamma(t) \cdot \dot{\mu}(t)$$

$$\text{Let } \gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t)), \quad \mu(t) = (\mu_1(t), \mu_2(t), \mu_3(t))$$

$$\frac{d}{dt}(\gamma(t) \cdot \mu(t)) = \frac{d}{dt}(\gamma_1 \mu_1 + \gamma_2 \mu_2 + \gamma_3 \mu_3)$$

$$= \dot{\gamma}_1 \mu_1 + \gamma_1 \dot{\mu}_1 + \dot{\gamma}_2 \mu_2 + \gamma_2 \dot{\mu}_2 + \dot{\gamma}_3 \mu_3 + \gamma_3 \dot{\mu}_3$$

$$= (\dot{\gamma}_1 \mu_1 + \dot{\gamma}_2 \mu_2 + \dot{\gamma}_3 \mu_3) + (\gamma_1 \dot{\mu}_1 + \gamma_2 \dot{\mu}_2 + \gamma_3 \dot{\mu}_3) = \dot{\gamma}(t) \cdot \mu(t) + \gamma(t) \cdot \dot{\mu}(t)$$

5. 8marks Prove that if $\gamma(t)$ is a unit-speed curve with curvature $K = 0$, then $\gamma(t)$ is contained in a line.

If $\gamma(t)$ is unit-speed, then $K = \|\ddot{\gamma}\|$

If $K = 0$, then $\|\ddot{\gamma}\| = 0 \Rightarrow \ddot{\gamma} = 0$

integrating once gives $\dot{\gamma} = \vec{a}$ for some vector $\vec{a}$

integrating again gives $\gamma(t) = \vec{a}t + \vec{b}$ for some vector $\vec{b}$

Thus, $\gamma(t)$ is contained in the line $\vec{a}t + \vec{b}$.

6.

- (a) **5marks** Sketch the curve $\gamma(t) = (\cos(t), \sin(t), t)$ where $t \in (-\infty, \infty)$ and determine its arc length for $t \geq 0$ starting from $\gamma(0)$.



helix

$$s(t) = \int_0^t \|\dot{\gamma}(u)\| du$$

$$\|\dot{\gamma}(u)\| = \|(-\sin u, \cos u, 1)\| = \sqrt{\sin^2 u + \cos^2 u + 1} = \sqrt{2}$$

$$s(t) = \int_0^t \sqrt{2} du = \sqrt{2} t$$

- (b) **6marks** Use your answer to part (a) to find a unit-speed reparametrization $\tilde{\gamma}(s)$ of the curve in (a).

we have $t = \frac{s}{\sqrt{2}}, s \geq 0$

$$\tilde{\gamma}(s) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right)$$

- (c) **9marks** Find the curvature K , torsion T , and the orthonormal vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ for the curve $\gamma(t)$ in (a).

$$\dot{\gamma}(t) = (-\sin t, \cos t, 1), \quad \ddot{\gamma}(t) = (-\cos t, -\sin t, 0), \quad \dddot{\gamma}(t) = (\sin t, -\cos t, 0)$$

$$K = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3}, \quad \dot{\gamma} \times \ddot{\gamma} = \det \begin{pmatrix} i & j & k \\ -s & c & 1 \\ -c & -s & 0 \end{pmatrix} = (\sin t, -\cos t, 1)$$

$$\|\dot{\gamma} \times \ddot{\gamma}\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$K = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{1}{2}, \quad T = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} = \frac{1}{(\sqrt{2})^2} = \frac{1}{2}$$

$$\vec{e}_1 = \frac{\dot{\gamma}}{\|\dot{\gamma}\|} = \frac{1}{\sqrt{2}} (-\sin t, \cos t, 1)$$

$$\vec{e}_2 = \frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|} = \frac{1}{\sqrt{2}} (\sin t, -\cos t, 1)$$

$$\vec{e}_3 = \vec{e}_1 \times \vec{e}_2 = \frac{1}{2} \det \begin{pmatrix} i & j & k \\ +s & -c & 1 \\ -s & c & 1 \end{pmatrix}$$

$$= \frac{1}{2} (-2c, -2s, 0) = (-\cos t, -\sin t, 0)$$

7.

- (a) **6marks** Define the *signed curvature* K_S of a planar curve in the x - y plane and compute it for the curve $\gamma(t) = (t, 1/t^2, 0)$, $t \in (0, \infty)$.

$$\frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma}\|^3} = K_S \hat{k} \quad \dot{\gamma} = \left(1, -\frac{2}{t^3}, 0\right) \quad \ddot{\gamma} = \left(0, \frac{6}{t^4}, 0\right)$$

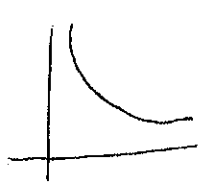
$$\dot{\gamma} \times \ddot{\gamma} = \frac{6}{t^4} \hat{k} \quad \|\dot{\gamma}\| = \left(1 + \frac{4}{t^6}\right)^{1/2}$$

$$K_S = \frac{6/t^4}{\left(1 + \frac{4}{t^6}\right)^{3/2}}$$

- (b) **6marks** State a result relating the angular change of a curve and an integral involving K_S . Use this with the curve in part (a) to show that $\int_0^\infty \frac{6t^2}{4+t^6} dt = \frac{\pi}{2}$.

angular change from $\gamma(t_0)$ to $\gamma(t_1)$

$$= \int_{t_0}^{t_1} K_S \|\dot{\gamma}\| dt$$



$$\tau = \frac{\dot{\gamma}}{\|\dot{\gamma}\|} = \frac{\left(1, -\frac{2}{t^3}, 0\right)}{\sqrt{1 + \frac{4}{t^6}}}$$

$$\tau \rightarrow (0, -1) \text{ as } t \rightarrow 0$$

$$\tau \rightarrow (1, 0) \text{ as } t \rightarrow \infty$$

$$\text{angular change} = \frac{\pi}{2}$$

$$\int_0^\infty \frac{6/t^4}{\left(1 + \frac{4}{t^6}\right)^{3/2}} dt = \int_0^\infty \frac{6/t^4}{1 + \frac{4}{t^6}} dt = \int_0^\infty \frac{6t^2}{t^6 + 4} dt = \frac{\pi}{2}$$

- (c) **8marks** If a planar curve $\gamma(t)$ in the x - y plane is unit-speed, then use the definition of K_S to show that $\ddot{\gamma} = K_S \vec{n}$ where $\vec{n} = \vec{k} \times \dot{\gamma}$ where $\vec{k}$ is the vector $\vec{k} = (0, 0, 1)$.

$$K_S \hat{k} = \dot{\gamma} \times \ddot{\gamma}$$

$$\begin{aligned} K_S \vec{n} &= K_S \hat{k} \times \dot{\gamma} = (\dot{\gamma} \wedge \ddot{\gamma}) \times \dot{\gamma} = \ddot{\gamma}(\dot{\gamma} \cdot \dot{\gamma}) - \dot{\gamma}(\dot{\gamma} \cdot \ddot{\gamma}) \\ &= \ddot{\gamma}(1) - \dot{\gamma}(0) \\ &= \ddot{\gamma} \end{aligned}$$